Introduction

FORMATIVE assessment is a key practice that, with effective implementation, positively affects students' achievement. A classroom environment that is conducive to effective formative assessment is one in which students

- feel safe to take risks while solving rich problems or tasks aligned with grade-level standards;
- are encouraged to work collaboratively;
- learn from one another and have the opportunity to share their work publicly as part of a community of discourse;
- have opportunities to reflect on and revise their mathematical writing and oral presentations to improve the clarity of their communication; and
- feel encouraged to work with each member of the community through the use of flexible grouping.

In such classrooms, teachers

- know the mathematics content and recognize key ideas and misconceptions;
- instruct students on appropriate protocols for working collaboratively in small groups;
- set clear expectations for student behaviors for individual and collaborative work;
- have the necessary tools to assess what their students know and can do;
- find ways to help students reflect on their problem solving, strategies, and writing to communicate with clarity; and
- know how to engage students in appropriate next steps effectively.

The description of the classroom environment that promotes effective formative assessment links back to the vision of school mathematics that opens *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM] 2000):

> Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. There are ambitious expectations for all, with accommodation for those who need it. Knowledgeable teachers have adequate resources to support their work and are continually growing as professionals. The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. Technology is an essential component of the environment. Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or dis-

> prove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it. (p. 3)

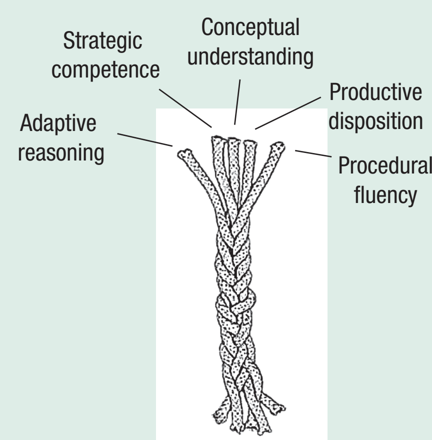
This vision of mathematics teaching and learning describes an interactive process in which classes explore mathematical ideas in a social context, where the communication of mathematical thinking is an integral part of learning, and where "students confidently engage in complex mathematical tasks chosen carefully by teachers" (NCTM 2000, p. 3). It implies a more complex definition of mathematical proficiency than simply the ability to compute fluently and manipulate mathematical expressions.

What Is Mathematical Proficiency?

The National Research Council's *Adding It Up: Helping Children Learn Mathematics* describes mathematical proficiency as five interconnected strands (Kilpatrick, Swafford, and Findell 2001):

1. Conceptual understanding
2. Procedural fluency
3. Strategic competence
4. Adaptive reasoning
5. Productive disposition

Intertwined Strands of Proficiency



The strands of mathematical proficiency build a picture of a mathematically proficient student:

- Students with *conceptual understanding* know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful.
- Students displaying *procedural fluency* know procedures and when to use them, and they can perform them flexibly, accurately, and efficiently.
- Students exhibiting *strategic competence* can formulate mathematical problems, represent them, and solve them.
- Students using *adaptive reasoning* can think logically about relationships among concepts and situations, consider alternatives, reason correctly, and justify conclusions.
- Students with a *productive disposition* see that mathematics makes sense and is both useful and worthwhile, believe that steady effort pays off, and see themselves as effective learners and doers of mathematics. (Kilpatrick, Swafford, and Findell 2001, chap. 4)

These strands—interwoven and interdependent—describe a set of knowledge, skills, abilities, and beliefs based on a body of research in cognitive psychology and mathematics education.

What Is Cognitive Demand?

Cognitive science studies how people learn. Levels of cognitive demand classify the kind of thinking that engaging with and solving a problem requires.

Until the middle of the twentieth century, the purpose of mathematics education for most people was to learn to compute accurately and efficiently. (Think of Bob Cratchit in Charles Dickens's *A Christmas Carol*, whom Ebenezer Scrooge employed to add long columns of numbers.) Elite institutions could produce the relatively few scientists and engineers needed for research and innovation—careers that required mathematical thinking. The emergence of computing technology during World War II, followed two decades later by the rapid growth of Asian science and engineering capabilities, has forced a long and highly emotional debate about the purpose of mathematics education and what mathematics is important for everyone to know and be able to do.

Part of the effort to define the characteristics of a new mathematics education system included the attempt to define exactly what kind of thinking children need to do as part of the learning process. Resnick described higher-order thinking in *Education and Learning to Think* (1987), when she proposed characteristics for the concept:

> Although we cannot define it exactly, we can recognize higher order thinking when it occurs. Consider the following:
>
> - Higher order thinking is *nonalgorithmic*. That is, the path of action is not fully specified in advance.
> - Higher order thinking tends to be *complex*. The total path is not "visible" (mentally speaking) from any single vantage point.
> - Higher order thinking often yields *multiple solutions*, each with costs and benefits, rather than unique solutions.
> - Higher order thinking involves *nuanced judgment* and interpretation.
> - Higher order thinking involves the application of *multiple criteria*, which sometimes conflict with one another.
> - Higher order thinking often involves *uncertainty*. Not everything that bears on the task at hand is known.
> - Higher order thinking involves *self-regulation* of the thinking process. We do not recognize higher order thinking in an individual when someone else "calls the plays" at every step.
> - Higher order thinking involves *imposing meaning*, finding structure in apparent disorder.
> - Higher order thinking is *effortful*. There is considerable mental work involved in the kinds of elaborations and judgments required. (pp. 2–3)

The Learning Research and Development Center at the University of Pittsburgh further developed these concepts in the early 1990s. The center sponsored the Quantitative Understanding: Amplifying Student Achievement and Reasoning (QUASAR) project with a grant from the Ford Foundation. QUASAR worked in Pittsburgh middle schools in a demonstration project aimed at raising low levels of student participation and performance in mathematics. They developed and implemented mathematics instructional programs based on three essential principles—that all students can (1) learn a broad range of mathematical content, (2) acquire a deeper and more meaningful understanding of mathematical ideas, and (3) demonstrate proficiency in mathematical reasoning and complex problem solving.

In the five years of the project, researchers developed the Mathematical Tasks Framework as a way to analyze classroom lessons. This framework identifies four levels of cognitive demand: memorization, procedures without connections, procedures with connections, and doing mathematics. Figure 1 describes each level.

Memorization tasks • involve either reproducing previously learned facts, rules, formulae, or definitions *or* committing facts, rules, formulae, or definitions to memory. • cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • are not ambiguous—such tasks involve exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated. • have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.	**Procedures with connections tasks** • focus students' attention on using procedures for developing deeper levels of understanding of mathematical concepts and ideas. • suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures to successfully complete the task and develop understanding.
Procedures without connections tasks • are algorithmic. Use of the procedure either is specifically called for or its use is evident based on prior instruction, experience, or placement of the task. • require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.	**Doing mathematics tasks** • requires complex and nonalgorithmic thinking (i.e., no predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example is evident). • requires students to explore and understand the nature of mathematical concepts, processes, or relationships. • demands self-monitoring or self-regulation of one's own cognitive processes.

Fig. 1. Characteristics of mathematical tasks at the four levels of cognitive demand—*Continues*

Procedures without connections tasks	Doing mathematics tasks
• have no connection to the concepts or meaning that underlie the procedure being used. • are focused on producing correct answers rather than developing mathematical understanding. • require no explanations or offer only explanations that focus solely on describing the procedure that was used.	• requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • requires students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • requires considerable cognitive effort and may involve some level of anxiety for the student owing to the unpredictable nature of the solution process required.

Fig. 1. Characteristics of mathematical tasks at the four levels of cognitive demand—*Continued*

"Doing mathematics tasks" is the highest level of cognitive demand and is closely related to the strands of strategic competence and adaptive reasoning that *Adding It Up* describes (Kilpatrick, Swafford, and Findell 2001). The NCTM Process Standards (NCTM 2000) further define the processes by which students "do math"—specific descriptions of the kinds of processes and habits of mind to integrate in our teaching to promote mathematical thinking that leads to proficiency (fig. 2).

Problem Solving

- Build new mathematical knowledge through problem solving.
- Solve problems that arise in mathematics and in other contexts.
- Apply and adapt a variety of appropriate strategies to solve problems.
- Monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof

- Recognize reasoning and proof as fundamental aspects of mathematics.
- Make and investigate mathematical conjectures.
- Develop and evaluate mathematical arguments and proofs.
- Select and use various types of reasoning and methods of proof.

Communication

- Organize and consolidate their mathematical thinking through communication.
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
- Analyze and evaluate the mathematical thinking and strategies of others.
- Use the language of mathematics to express mathematical ideas precisely.

Fig. 2. NCTM Process Standards—*Continues*

Connections

- Recognize and use connections among mathematical ideas.
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
- Recognize and apply mathematics in contexts outside mathematics.

Representation

- Create and use representations to organize, record, and communicate mathematical ideas.
- Select, apply, and translate among mathematical representations to solve problems.
- Use representations to model and interpret physical, social, and mathematical phenomena.

Fig. 2. NCTM Process Standards—*Continued*

Teaching for Mathematical Proficiency

Resnick noted in the conclusion to her 1987 work,

> Thinking skills tend to be driven out of the curriculum by ever-growing demands for teaching larger and larger bodies of knowledge. The idea that knowledge must be acquired first and that its application to reasoning and problem solving can be delayed is a persistent one in educational thinking. “Hierarchies” of educational objectives, although intended to promote attention to higher order skills, paradoxically feed this belief by suggesting that knowledge acquisition is a first stage in a sequence of educational goals. The relative ease of assessing people’s knowledge, as opposed to their thought processes, further feeds this tendency in educational practice. (pp. 48–49)

More than twenty years later, we still struggle to change an entrenched, traditional view of mathematics education and assessment that typically focuses on memorization and procedures without connections. We have all had the experience of teaching a mathematical procedure one day, being fairly certain that the lesson was successful and that most students could perform the procedure at the end of it, and realizing later that many of those same students have forgotten what they learned. Mathematical proficiency will not result from continual procedural instruction, nor will we know what kind of thinking students can do if we assess only their procedural knowledge.

Assessing Mathematical Proficiency

We are all familiar with summative assessments: the typical “math test” with problems that are easily marked correct or incorrect, that result in a grade, or that rank students in relation to other students.

However, we tend to be less familiar with formative assessments. Formative assessment is assessment *for* learning, whereas summative assessment is *of* learning. Formative assessments make students' thinking visible.

To provide learning experiences that build on and increase students' understanding—and to develop and deepen students' ability to reason and communicate mathematically—teachers must know what their students are thinking. Evidence from formative assessment allows the teacher to delve beneath students' factual knowledge to probe their depth of understanding. Formative assessment offers evidence of student learning that teachers can use to make informed decisions about the next question to ask and the next problem to assign or to determine which students to group together for the next mathematical task.

For instance, a student who demonstrated mastery in finding equivalent fractions was asked to name two fractions that come between $^3/_5$ and $^4/_5$. The student responded, "There are no fractions between 3 and 4." It appeared that the student understood equivalent fractions on the basis of a correct response to a summative question. But the response to the follow-up question, at a higher level of cognitive demand, indicated the student's limited understanding of fractions.

Exposing the depth of the student's understanding took only one good question—and this is the potential power of formative assessment. With the additional knowledge gathered from questions that cannot be answered by using a memorized fact or procedure, the teacher can differentiate instruction in the next lesson to extend each student's conceptual understanding of fractions and equivalent fractions with tasks that require more student thinking.

Overview of This Book

Chapter 1 includes general formative assessment information and strategies. Chapter 2 offers specific activities, protocols, and strategies for gathering evidence to help teachers make informed decisions about where in the learning progression their students are functioning. It will suggest ways to move students along their learning trajectory to attain the learning standards. Chapters 3–5 include formative assessment items, strategies, and protocols for the key mathematical ideas and connections from *Curriculum Focal Points: A Quest for Coherence* (NCTM 2006), all aligned with the Common Core State Standards for grades 6–8. The professional development guide outlines structures for workshops that can help teachers effectively use formative assessment in the mathematics classroom.

References

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